

INVOLUTIONS OF SECOND KIND ON SHIMURA SURFACES AND SURFACES OF GENERAL TYPE WITH

$$q = p_g = 0$$

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ABSTRACT. Quaternionic Shimura surfaces are quotient of the bidisc by an irreducible cocompact arithmetic group. In the present paper we are interested in (smooth) quaternionic Shimura surfaces admitting an automorphism with one dimensional fixed locus; such automorphisms are involutions. We propose a new construction of surfaces of general type with $q = p_g = 0$ as quotients of quaternionic Shimura surfaces by such involutions. These quotients have finite fundamental group.

1. INTRODUCTION

Among smooth minimal surfaces of general type, the ones with vanishing geometric genus p_g are of main interest (see e.g. [BFP11]). For such surfaces, the Chern number $c_1^2 = K^2$ belongs to the set $\{1, \dots, 9\}$ and $c_2 = 12 - c_1^2$. We are far away from a complete classification, although great advances have been done recently, e.g. for surfaces with $c_1^2 = 9$, the fake projective planes, which have been completely classified, see [PY07], [CS10]. In the other cases, a major task is to construct new examples of such surfaces.

In this paper we give a uniform construction of surfaces with $q = p_g = 0$ and $c_1^2 = 1, \dots, 7$. These surfaces are obtained as quotients of smooth quaternionic Shimura surfaces X by a special kind of involution. Recall that a smooth Shimura surface $X = X_\Gamma$ is the quotient of $\mathbb{H} \times \mathbb{H}$, the product of two copies of the complex upper half plane, by a discrete cocompact torsion free group $\Gamma \subset \text{Aut}(\mathbb{H}) \times \text{Aut}(\mathbb{H})$ of holomorphic automorphisms defined by certain quaternion algebra. The invariants of X are $c_1^2(X) = 2c_2(X) = 8(1 + p_g(X))$ and $q(X) = 0$.

In the first step we prove that an automorphism of a Shimura surface is quite special:

Theorem 1. *Let σ be an automorphism of a smooth Shimura surface. Then either σ has a finite number of fixed points or the fixed point set of σ is a divisor and $\sigma^2 = 1$.*

Under certain conditions on the group Γ , the involution $\mu \in \text{Aut}(\mathbb{H} \times \mathbb{H})$ exchanging the two factors induces an involution σ on the surface X . We call such an automorphism of X an involution of second kind. We obtain

Theorem 2. *Let X be a smooth Shimura surface admitting an involution of second kind σ . The fixed point set C of σ is a union of disjoint smooth Shimura curves. The arithmetic genus g of C satisfies $2 \leq g \leq p_g(X)$ and the quotient surface $Z = X/\sigma$ is smooth with finite fundamental group. If moreover $g = p_g(X) \leq 8$, Z is a surface of general type with invariants:*

$$c_1(Z)^2 = 9 - p_g(X), \quad c_2(Z) = 3 + p_g(X), \quad p_g(Z) = q(Z) = 0.$$

If $p_g = 2$ or 3 , one can prove that $g = p_g$ and therefore $c_1^2 = 7$ and 6 respectively.

We then concentrate on the construction of such Shimura surfaces with low p_g and admitting an involution of second kind. It turns out that such surfaces are rather exceptional. For instance, if we restrict our consideration to totally real fields of degree 2, there are at most 14 isomorphism classes of quaternion algebras leading to smooth Shimura surfaces of geometric genus $2 \leq p_g \leq 8$ admitting an involution of second kind (see Theorem 22). On the other hand we consider Shimura surfaces corresponding to congruence subgroups and we are able to show:

Theorem 3. *For $k = 5, 6$ there exists a smooth Shimura surface X with an involution of second kind σ and $p_g(X) = k$. In the case $k = 5$, the curve C fixed by σ is irreducible of genus $g(C) = 5 = p_g(X)$.*

In the light of some open questions concerning fundamental groups of surfaces with geometric genus zero, see [BFP11], an example of a smooth Shimura surface with $p_g = 2$ admitting an involution of second kind would be highly interesting. However, finding such a surface turns out to be very difficult (see Remark 30).

This paper mixes two fields: Theory of Shimura surfaces and classical algebraic geometry of surfaces. Shimura surfaces are closely related to Hilbert modular surfaces (which were first systematically studied by Hirzebruch, see for instance [vdG88], but are less known and studied. It was also one of our aim to develop that theory of Shimura surfaces.

The paper is organized as follows: In Section 2 we discuss the conditions on Γ under which the Shimura surface X_Γ has an involution of second kind. In Section 3 we study the quotient surface of a smooth Shimura surface by the action of an involution of second kind and we prove that its fundamental group is finite. In Section 4, we investigate examples of Shimura surfaces with low geometric genus admitting an involution of second kind. In particular we develop new tools to create smooth quaternionic Shimura surfaces and we make a systematic study of Shimura surfaces defined over quadratic fields with low geometric genus. We give examples of surfaces with $p_g = 5, 6$ admitting an involution of second kind. Finally, in section 5 we present the method to identify the Shimura curve fixed by an involution of second kind acting on a Shimura surface and we furthermore examine the example with $p_g = 5$.

2. INVOLUTIONS OF SECOND KIND ACTING ON SHIMURA SURFACES

2.1. Quaternionic Shimura surfaces. Let us recall the construction of quaternionic Shimura surfaces.

Let k be a totally real number field of degree $n = [k : \mathbb{Q}]$. The places of k are the equivalence classes of valuations on k , and the infinite places of k correspond to embeddings $\sigma_i \in \text{Hom}_{\mathbb{Q}}(k, \mathbb{R})$, $i = 1, \dots, n$.

Let A be a division quaternion algebra whose center is k . For every place v of k , we denote by k_v the completion of k with respect to v and define $A_v = A \otimes_k k_v$. The algebra A is *ramified* at v if A_v is a division algebra over k_v and *unramified* otherwise, that is, if $A_v \cong M_2(k_v)$. By the classical theorem of Hasse and the product formula for Hilbert symbols, the isomorphism class of A is uniquely determined by the set $\text{Ram}(A)$ of ramified places of A . Assume that A is unramified at the first, say $m \leq n$, infinite places $\sigma_1, \dots, \sigma_m$ and ramified at the remaining $n - m$ infinite places. Equivalently we assume that we have an isomorphism

$$A \otimes_{\mathbb{Q}} \mathbb{R} \rightarrow M_2(\mathbb{R})^m \times \mathbf{H}^{n-m}$$

where \mathbf{H} denotes the skew field of Hamiltonian quaternions. Then, A is uniquely determined up to an isomorphism by the tuple

$$(k, \sigma_1, \dots, \sigma_m, \mathfrak{p}_1, \dots, \mathfrak{p}_r),$$

where $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ are the non-archimedean places of k where A is ramified, and $\sigma_1, \dots, \sigma_m$ are the infinite places of k such that $A \otimes_{\sigma_i(k)} \mathbb{R} \cong M_2(\mathbb{R})$. Since we are interested in algebraic surfaces, we suppose that there are exactly two infinite places σ_1, σ_2 such that A is unramified at the σ_i . With this fixed ramification behaviour at the infinite places, the isomorphism class of A depends only on the tuple $(\mathfrak{p}_1, \dots, \mathfrak{p}_r)$. Let us write

$$A = A(k, \mathfrak{p}_1, \dots, \mathfrak{p}_r),$$

(omitting the explicit indication of two unramified infinite places) for such a quaternion algebra in the following. The subgroup A^+ of A consisting of the units of A having totally positive reduced norm can be identified via the isomorphism $A \otimes_{\mathbb{Q}} \mathbb{R} \xrightarrow{f} M_2(\mathbb{R})^2 \times \mathbf{H}^{n-2}$ with a subgroup of $GL_2^+(\mathbb{R})^2 \times \mathbf{H}^{*n-2}$, and projecting to the first two factors gives an injection of A^+ into $GL_2^+(\mathbb{R})^2$. We denote by

$$\rho = (\rho_1, \rho_2) : A \rightarrow M_2(\mathbb{R})^2$$

this representation of A . Note that these ρ_i , $i = 1, 2$ are extensions of two morphisms $k \rightarrow \mathbb{R}$ (where $\mathbb{R} \subset M_2(\mathbb{R})$ is identified with diagonal matrices) corresponding to the places σ_1, σ_2 .

Let us denote by \mathcal{O} and \mathcal{O}_k a maximal order of A and the ring of integers of k . Let \mathcal{O}^* denote the group of units of \mathcal{O} , \mathcal{O}^+ the group of units in \mathcal{O} with totally positive reduced norm and $\mathcal{O}^1 \subset \mathcal{O}$ the group of units of reduced norm 1. The group \mathcal{O}^1 is via ρ a discrete subgroup of $SL_2(\mathbb{R})^2$, whereas \mathcal{O}^+ is embedded as a discrete subgroup in $GL_2^+(\mathbb{R})^2$.

The group A^+ and any subgroup $G \subset A^+$ acts on the product $\mathbb{H} \times \mathbb{H}$ of two copies of the upper half plane as follows: if $(z, w) \in \mathbb{H} \times \mathbb{H}$ is a point and $g \in G$, then $\rho(g)$ is represented by two matrices $g_i = \rho_i(g) = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}$ ($i = 1, 2$) and:

$$g(z, w) = (g_1 z, g_2 w) := \left(\frac{a_1 z + b_1}{c_1 z + d_1}, \frac{a_2 w + b_2}{c_2 w + d_2} \right),$$

where a_1, a_2 (resp. $(b_1, b_2) \dots$) are conjugates with respect to the places σ_1, σ_2 over the Galois closure of k in \mathbb{R} . The action of G is not effective; the center $Z(G)$ acts trivially on $\mathbb{H} \times \mathbb{H}$ and therefore we will consider subgroups $\Gamma = G/Z(G) \subset A^+/k^*$ rather than subgroups $G \subset A^+$. Let us write $\Gamma_{\mathcal{O}}(1)$ or sometimes simply $\Gamma(1)$ for the group $\mathcal{O}^1/\{\pm 1\}$. The group $\Gamma(1)$ and also any subgroup Γ of A^+/k^* commensurable with $\Gamma(1)$ acts properly discontinuously on $\mathbb{H} \times \mathbb{H}$.

The quotient $X_\Gamma = \mathbb{H} \times \mathbb{H}/\Gamma$ is a compact algebraic surface, which will be called *quaternionic Shimura surface* (corresponding to Γ) in the sequel.

2.2. The involution exchanging factors and involutions of the second kind. Let, as above, k be a totally real number field and consider the quaternion algebra $A = A(k, \mathfrak{p}_1, \dots, \mathfrak{p}_r)$ over k unramified at two infinite places of k and ramified at all the other infinite places of k and at the finite places $\mathfrak{p}_1, \dots, \mathfrak{p}_r$.

Definition 4. An *involution of second kind* on A is a map $\tau : A \rightarrow A$ such that $\tau^2(a) = a$, $\tau(a + b) = \tau(a) + \tau(b)$, $\tau(ab) = \tau(a)\tau(b)$ for all $a, b \in A$ and such that the restriction of τ to k is a non-trivial automorphism of k .

Let $\ell = k^\tau$ be the fixed field of τ . Then k/ℓ is a quadratic extension and in this case we will say that τ is a *k/ℓ -involution*.

Remark 5. (See also [Gra02, Lemma 4.2]). Let τ and σ be two k/ℓ -involutions of second kind on A . Then there exists $m \in A^*$ such that $\sigma(a) = m^{-1}\tau(a)m$ and $\tau(m)^* = m$, where $a \rightarrow a^*$ is the canonical anti-involution on A , that is, the uniquely determined anti-involution $*$: $A \rightarrow A$ such that the reduced norm is $Nrd(x) = xx^*$ and the reduced trace is $Trd(x) = x + x^*$.

Following [Gra02] we choose to work with involutions on A which are, by above definition, particularly ring homomorphisms of A . In the literature more often one works with anti-involutions, that is, with maps $\rho : A \rightarrow A$ satisfying $\rho(ab) = \rho(b)\rho(a)$. These two kinds of maps are linked by the canonical anti-involution. First observe that every involution of second kind commutes with the canonical anti-involution, that is, for every $a \in A$ we have $\tau(a)^* = \tau(a^*)$, see [Gra02]. From this it follows that there is one-to-one correspondence between involutions and anti-involutions on A given by the rule

$$\tau \mapsto \tau^* \text{ and } \rho \mapsto \rho^*$$

where τ^* , resp. ρ^* , is defined as $\tau^*(a) = \tau(a)^*$, resp. $\rho^*(a) = \rho(a)^*$, for every $a \in A$. In this way, involutions of second kind correspond to classically studied anti-involutions of second kind.

A criterion for the existence of anti-involutions of second kind is well-known and goes back to work of Albert and Landherr.

Proposition 6. *(See also [Gra02, Lemma 4.3] and more generally [Lan37, Theorem 3]). Let k/ℓ be a quadratic extension of totally real fields. Let α be the non-trivial ℓ -automorphism of the extension k/ℓ and let $A = A(k, \mathfrak{p}_1, \dots, \mathfrak{p}_r)$. Then, there exists a k/ℓ -involution τ of second kind on A (i. e. $\tau(xa) = x^\alpha \tau(a)$ for all $x \in k, a \in A$) if and only if*

- (1) *r is even and after a suitable renumbering of the \mathfrak{p}_i , we have $\mathfrak{p}_{2i-1} = \mathfrak{p}_{2i}^\alpha$ and $\mathfrak{p}_i \neq \mathfrak{p}_i^\alpha$ ($i = 1, \dots, r/2$).*
- (2) *With the notations and hypothesis of section 2.1, it holds that $\sigma_2 = \sigma_1 \circ \alpha$.*

Proof. Let us first show that there exists an k/ℓ -involution τ on A if and only if there exists a quaternion subalgebra $A' \subset A$ whose center is ℓ .

Namely, if such A' exists then $A' \otimes_\ell k = A$ and $a \otimes x \mapsto a \otimes x^\alpha$ is a k/ℓ -involution. Conversely, let $A' = \{a \in A \mid \tau(a) = a\}$. This is a ℓ -subalgebra of A . Let $\theta \in k$ be such that $k = \ell(\theta)$ and $\tau(\theta) = -\theta$. We claim that $A = A' \oplus \theta A'$. To see this we write an element $a \in A$ as $a = \frac{1}{2}(a + \tau(a)) + \frac{1}{2}\theta \frac{a - \tau(a)}{\theta}$. It follows that $A = A' + \theta A'$. Clearly, an element $a \in A' \cap \theta A'$ satisfies $\tau(a) = a = -a$, hence $a = 0$ and it follows that $kA' \cong k \otimes_\ell A' = A$ and therefore A' is a quaternion algebra over ℓ since $A = A' \otimes_\ell k$ is a quaternion algebra.

Now we compare the sets of ramification of A and A' . If v is a place of ℓ such that $A' \otimes_\ell \ell_v \cong M_2(\ell_v)$, then for any place w of k lying over v we have $A \otimes_k k_w \cong M_2(k_w)$, so if A' is unramified at a place v of ℓ , then A is unramified at every place w of k lying over v . Particularly, since k and ℓ are totally real, if v is an infinite place of ℓ , that is, an embedding $v : \ell \hookrightarrow \mathbb{R}$ then there are exactly two places of k lying over v , that is, two embeddings $w_1, w_2 : k \hookrightarrow \mathbb{R}$ extending v and satisfying $w_2 = w_1 \circ \alpha$.

Assume now that v is a place of ℓ where A' is ramified. If v is an infinite place of ℓ , then, again since k and ℓ are both totally real, A is ramified at every place w of k over v . Assume that v is a finite place of ℓ corresponding to a prime ideal \mathcal{P} . There are two different possibilities: \mathcal{P} is split in k , that is, $\mathcal{P}\mathcal{O}_k = \mathfrak{p}\mathfrak{p}^\alpha$, with two prime ideals $\mathfrak{p} \neq \mathfrak{p}^\alpha$ of k . As $k_{\mathfrak{p}} = \ell_{\mathcal{P}}$, A' is ramified at \mathcal{P} if and only if A is ramified at \mathfrak{p} . If \mathcal{P} is non-split in k , i. e., if only one prime \mathfrak{p} is lying over \mathcal{P} , then by the theorem of Hasse, the field $k_{\mathfrak{p}}$ can be embedded into $A' \otimes_\ell \ell_{\mathcal{P}}$ and therefore $A' \otimes_\ell \ell_{\mathcal{P}} \cong M_2(k_{\mathfrak{p}})$. In this case A is unramified at \mathfrak{p} . It follows, that the existence of a k/ℓ -involution implies the conditions (1) and (2).

Conversely, choosing the set of ramified places according to (1) and (2), we can construct a quaternion algebra A' over ℓ such that $A' \otimes_\ell k$ and A

are ramified exactly at the same set of primes. This implies an isomorphism $A \cong A' \otimes_{\ell} k$ and thus the existence of an k/ℓ -involution of second kind \square

Remark 7. Let us note that with notations of section 2.1 the above proposition implies that in the case of the existence of a k/ℓ -involution τ on A we particularly have the relation $\rho_2 = \rho_1 \circ \tau$ on A . Note also that the condition (2) is superfluous in the case where k is a real quadratic field, since there is only one choice of $\sigma_2 = \sigma_1 \circ \alpha$.

Let τ be an involution of second kind on A . Let $\Gamma \subset A^+/k^*$ be a subgroup stable by τ , that is, for all $\gamma \in \Gamma$ we have $\tau(\gamma) \in \Gamma$. Since $\rho_2 = \rho_1 \circ \tau$, the images of Γ in $PSL_2(\mathbb{R})$ by ρ_1 and ρ_2 are the same, and since this image is isomorphic to Γ we denote it also by Γ . In order to avoid more long-winded notations, let us write for an involution of second kind τ shortly

$$\tau(a) = \bar{a} \text{ for all } a \in A.$$

In particular, the non-trivial ℓ -automorphism of k is denoted by $x \mapsto \bar{x}$ (for $x \in k$). Identifying k with $\sigma_1(k)$, say, the action of any element $\gamma \in A^+/k^*$ on $\mathbb{H} \times \mathbb{H}$ given by

$$\gamma(z_1, z_2) = (\rho_1(\gamma)z_1, \rho_2(\gamma)z_2)$$

is written as

$$\gamma(z_1, z_2) = (\gamma z_1, \bar{\gamma} z_2).$$

Let $\mu : \mathbb{H} \times \mathbb{H} \rightarrow \mathbb{H} \times \mathbb{H}$ be the involution that exchanges the two factors. The group $Aut(\mathbb{H} \times \mathbb{H})$ is the semi direct product of $Aut(\mathbb{H}) \times Aut(\mathbb{H})$ and the group $\mathbb{Z}/2\mathbb{Z}$ generated by μ .

Proposition 8. *Let Γ be a subgroup of A^+ commensurable with \mathcal{O}^1 for some maximal order \mathcal{O} in A . Suppose that there exists an involution τ of second kind on A preserving Γ . Then, the automorphism μ of $\mathbb{H} \times \mathbb{H}$ induces an involution σ on the surface $X_{\Gamma} = \mathbb{H} \times \mathbb{H}/\Gamma$.*

Proof. We need to prove that μ sends an orbit under the action of Γ to another Γ -orbit. We have:

$$\Gamma(z_1, z_2) = \{(\gamma z_1, \bar{\gamma} z_2) \mid \gamma \in \Gamma\},$$

thus

$$\mu(\Gamma(z_1, z_2)) = \{(\bar{\gamma} z_2, \gamma z_1) \mid \gamma \in \Gamma\}.$$

Since $\tau : a \rightarrow \bar{a}$ preserves Γ , we get:

$$\mu(\Gamma(z_1, z_2)) = \{(\gamma z_2, \bar{\gamma} z_1) \mid \gamma \in \Gamma\} = \Gamma(z_2, z_1),$$

therefore $\mu(\Gamma(z_1, z_2)) = \Gamma(z_2, z_1)$ is an orbit under the action of Γ on $\mathbb{H} \times \mathbb{H}$ and μ acts on the orbit space X_{Γ} . \square

2.3. Automorphisms of Shimura surfaces. Let $\mu \in \text{Aut} \mathbb{H} \times \mathbb{H}$ be the involution that exchanges the two factors. Let $X_\Gamma = \mathbb{H} \times \mathbb{H} / \Gamma$ be a smooth quaternionic Shimura surface. By [DR14, Theorem 3.12] and its proof, we get:

Proposition 9. *Suppose that the fixed locus C of an automorphism σ of $X = X_\Gamma$ contains an one-dimensional component. Then σ is an involution that lifts on the universal cover to (a conjugate of) μ .*

Let us consider an involution $a \mapsto \bar{a}$ on A of second kind and a torsion-free group Γ commensurable with a group $\Gamma(1)$, and as above stable under $a \mapsto \bar{a}$. Recall that for $\gamma \in \Gamma$ and $t = (z_1, z_2) \in \mathbb{H} \times \mathbb{H}$ the action of γ on $\mathbb{H} \times \mathbb{H}$ is given by

$$\gamma t = (\gamma z_1, \bar{\gamma} z_2).$$

The image of t on $X_\Gamma = \mathbb{H} \times \mathbb{H} / \Gamma$ is a fixed point of the involution σ induced by μ if and only if

$$\Gamma(z_1, z_2) = \Gamma(z_2, z_1),$$

which is the case if and only if there exists γ in Γ such that

$$(z_1, z_2) = \gamma(z_2, z_1) = (\gamma z_2, \bar{\gamma} z_1),$$

that is, if and only if $z_1 = \gamma z_2$ and $z_2 = \bar{\gamma} \gamma z_2$. Since Γ is torsion-free (because X is smooth), the image of t in X is a fixed point of σ if and only if $\bar{\gamma} \gamma = 1$.

As in [Gra02], for any $\beta \in A \setminus \{0\}$, let Δ_β be the disk $\Delta_\beta = \{(z, \beta z) / z \in \mathbb{H}\}$. We have $\lambda \Delta_\beta = \Delta_{\bar{\lambda} \beta \lambda^{-1}}$ for any $\lambda \in A \setminus \{0\}$. We denote by F_β the image of Δ_β in X_Γ .

Let $\gamma \in \Gamma$ be such that $\bar{\gamma} = \gamma^{-1}$. Since $\bar{\gamma} \gamma = 1$, we have $(\gamma z, z) = \gamma(z, \gamma z)$, therefore for any point t of Δ_γ , we obtain

$$\mu \Gamma t = \Gamma \mu t = \Gamma \gamma t = \Gamma t,$$

and the image F_γ of Δ_γ on X_Γ is fixed by the involution σ . That implies that F_γ is a smooth irreducible algebraic curve, more precisely a Shimura curve, and that there is only a finite number of such curves. We obtain:

Corollary 10. *The fixed point set of σ is the union of smooth disjoint Shimura curves F_γ with $\gamma \in \Gamma$ such that $\gamma \bar{\gamma} = 1$.*

Remark 11. Since the irreducible components of the fixed locus C are smooth disjoint Shimura curves on the Shimura surface X , we get by the Hirzebruch Proportionality Theorem

$$2C^2 = -K_X C = 4(1 - g)$$

where $g > 1$ is the arithmetic genus of C . If the irreducible components are C_j , $j = 1 \dots k$ of genus g_j , then $C_j^2 = 2(1 - g_j)$ and $C^2 = \sum C_j^2 = \sum 2(1 - g_j)$. Moreover

$$K_X C = \sum 4(g_j - 1)$$

thus $C^2 + K_X C = \sum 2g_j - 2 = 2g - 2$ and $C^2 = 2 - 2g$.

Recall that $\lambda C_\beta = C_{\bar{\lambda}\beta\lambda^{-1}}$ for any $\lambda \in A \setminus \{0\}$, and in particular for every $\lambda \in \Gamma$ we have:

$$F_1 = F_{\bar{\lambda}\lambda^{-1}}.$$

Of course, for $\alpha = \bar{\lambda}\lambda^{-1}$ we have $\alpha\bar{\alpha} = 1$. We conjecture that for a smooth surface X the fixed locus of such σ is irreducible. As we see immediatly, this is equivalent to the following:

Conjecture 12. *Let be $\lambda \in \Gamma$ such that $\lambda\bar{\lambda} = 1$. There exists $\gamma \in \Gamma$ such that $\lambda = \bar{\gamma}\gamma^{-1}$.*

3. QUOTIENT OF A QUATERNIONIC SHIMURA SURFACE BY AN INVOLUTION OF SECOND KIND.

3.1. Invariants of the quotient. Let Γ be a lattice preserved by an involution of the second kind and let σ be the corresponding involution acting on the Shimura surface $X = X_\Gamma$. Let C be the smooth curve of arithmetic genus g fixed by the involution.

Proposition 13. *The quotient surface $Z = X/\sigma$ is smooth and has invariants:*

$$K_Z^2 = e(X) + 5(1 - g), \quad c_2 = \frac{1}{2}e(X) + 1 - g, \quad p_g = \frac{e(X) - 4 - 4g}{8}, \quad q = 0,$$

where $e(X) = c_2(X)$ is the topological Euler number of X .

If $(K_X - C)^2 > 0$, then Z has general type; this condition on the positivity is satisfied if $e(X) \leq 36$.

Suppose $e(X) = 12$, then C is irreducible of genus $g = 2$ and the quotient surface X/σ has invariants:

$$K_X^2 = 7, \quad c_2 = 5, \quad p_g = 0, \quad q = 0.$$

Suppose $e(X) = 16$, then C is irreducible of genus $g = 3$ and the quotient surface Z has invariants:

$$K_X^2 = 6, \quad c_2 = 6, \quad p_g = 0, \quad q = 0.$$

Proof. Let $\pi : X \rightarrow Z = X/\sigma$ be the quotient map. Since $K_X^2 = 2e$ (for $e = c_2(X)$), $K_X = \pi^*K_Z + C$ and $C^2 = 2(1 - g)$, $K_X C = 4(g - 1)$ (here we use that C is a disjoint union of smooth Shimura curves) we get

$$K_Z^2 = \frac{1}{2}(K_X - C)^2 = \frac{1}{2}(K_X^2 - 2K_X C + C^2) = e - 5(g - 1).$$

Moreover by general formulas on quotient surfaces (see e.g. [DR14])

$$e(Z) = \frac{1}{2}(e(X) + e(C)) = \frac{1}{2}e(X) + 1 - g.$$

As $q(X) = 0$, we get $q(Z) = 0$ and

$$p_g(Z) = \chi(\mathcal{O}_Z) - 1 = \frac{e(X) - 4(g + 1)}{8}.$$

As $g \geq 2$ because X is hyperbolic and $p_g(Z) \geq 0$, we get that

$$2 \leq g \leq \frac{e(X) - 4}{4},$$

thus

$$e(Z) \geq \frac{e(X)}{4} + 2, \chi(\mathcal{O}_Z) \geq 1$$

and $K_Z^2 \geq 10 - \frac{e}{4}$. Let us prove that Z has general type if $K_Z^2 > 0$. Since $\pi^*K_Z = K_X - C$, it is enough to prove that powers \mathcal{L}^m of $\mathcal{L} = \mathcal{O}_X(K_X - C)$ have sections growing in $c \cdot m^2$, where $c > 0$ is a constant. Suppose that $K_Z^2 = \frac{1}{2}(K_X - C)^2 > 0$ (note that by the preceding discussion, this condition is always satisfied for a surface X with $e(X) < 40$). By the Riemann-Roch Theorem, we have

$$\chi(\mathcal{L}^m) = \frac{m^2}{2}(K_X - C)^2 - \frac{m}{2}K_X(K_X - C) + \chi(\mathcal{O}_X).$$

Serre duality gives

$$\chi(\mathcal{L}^m) = H^0(X, \mathcal{L}^m) - H^1(X, \mathcal{L}^m) + H^0(X, mC - (m-1)K_X).$$

Suppose that $D = mC - (m-1)K_X$ is effective. As K_X is ample $K_X D > 0$. But

$$K_X D = m(4g - 4 - 2e) + 2e$$

and as $g \leq \frac{e(X)-4}{4}$, we get $4g - 4 \leq e - 8$ and

$$K_X D \leq m(-8 - e) + 2e < 0$$

for $m \geq 3$. Therefore $H^0(X, mC - (m-1)K_X) = 0$ and Z has general type. \square

The possibilities for values 12, 16, \dots , 36 of $e(X)$ and for the genus g of C are listed in the above tables:

$e(X)$	g	K_Z^2	$c_2(Z)$	$p_g(Z)$	$e(X)$	g	K_Z^2	$c_2(Z)$	$p_g(Z)$
$e = 12$	2	7	5	0	$e = 28$	6	3	9	0
$e = 16$	3	6	6	0	$e = 32$	3	22	14	2
$e = 20$	2	15	9	1		5	12	12	1
	4	5	7	0		7	2	10	0
$e = 24$	3	14	10	1	$e = 36$	2	31	17	3
	5	4	8	0		4	21	15	2
$e = 28$	2	23	13	2		6	11	13	1
	4	13	11	1		8	1	11	0

3.2. The fundamental group of the quotient. Let us recall some results about fundamental groups. Let G be a discontinuous group of homeomorphisms of a path connected, simply connected, locally compact metric space M , and let G_{tor} be the normal subgroup of G generated by those elements which have fixed points, or equivalently, the torsion elements. Then

Theorem 14. [Arm68]. *The fundamental group of the orbit space M/G is isomorphic to the factor group G/G_{tor} .*

Let $X = X_\Gamma$ be our Shimura surface with fundamental group Γ such that the involution μ switching the two factors of the bi-disk $\mathbb{H} \times \mathbb{H}$ acts on X by an involution denoted by σ . The fundamental group of the quotient surface X/σ is isomorphic to $\Gamma'/\Gamma'_{\text{tor}}$, where Γ' is the group generated by Γ and μ .

Recall that a group G acting on the space M is discontinuous if:

- (1) the stabilizer of each point of M is finite, and
- (2) each point $x \in M$ has a neighborhood U such that any element of G not in the stabilizer of x maps U outside itself (i.e. if for $g \in G$, $gx \neq x$ then $U \cap gU$ is empty).

Lemma 15. *The group Γ' is discontinuous.*

Proof. Since $M = \mathbb{H} \times \mathbb{H}$ is a locally compact Hausdorff space, Γ' is discontinuous if and only if it is discrete subgroup of $\text{Aut}(\mathbb{H} \times \mathbb{H})$. The latter assertion follows from the fact that Γ' is an index-2 extension of the discrete group Γ . \square

For any $\gamma \in \Gamma$, we have $\mu\gamma = \bar{\gamma}\mu$. Let $g \in \Gamma'_{\text{tor}}$ be a non-trivial torsion element. Since Γ is torsion free, $g \notin \Gamma$ and there exists $\lambda \in \Gamma$ such that $g = \lambda\mu$. The order of g is then divisible by 2 and we have $g^{2n} = (\lambda\bar{\lambda})^n$. Since Γ is torsion free, $g^{2n} = 1$ if and only if $\lambda\bar{\lambda} = 1$. Therefore a torsion element g of Γ' has order 2 and there exists $\lambda \in \Gamma$ such that $g = \lambda\mu$ and $\lambda\bar{\lambda} = 1$.

As an immediate consequence we obtain:

Lemma 16. *The fundamental group $\Gamma'/\Gamma'_{\text{tor}}$ of X/σ is isomorphic to the group Γ/N where N is the normal group generated by the $\lambda \in \Gamma$ such that $\lambda\bar{\lambda} = 1$.*

Since for any $\gamma \in \Gamma$, the group Γ'_{tor} contains $\bar{\gamma}\mu\bar{\gamma}^{-1} = \bar{\gamma}\gamma^{-1}\mu$, we see that N contains $\bar{\gamma}\gamma^{-1}$ for every $\gamma \in \Gamma$, therefore the quotient Γ/N forces the relation $\gamma = \bar{\gamma}$ for any $\gamma \in \Gamma$.

Let us denote by H the normal subgroup of Γ generated by the elements $\bar{\gamma}\gamma^{-1}$, $\gamma \in \Gamma$. Note that under Conjecture 12, the group H is equal to N . The group $\pi_1(X/\sigma) = \Gamma'/\Gamma'_{\text{tor}} \simeq \Gamma/N$ is a quotient of Γ/H .

Theorem 17. *Suppose that Γ is a subgroup of $\Gamma(1)$. Then Γ/H is a finite group and the fundamental group of X/σ is finite.*

Proof. Let A' be a quaternion algebra over the field ℓ as above such that $A = A' \otimes k$ and the involution of second kind on A is given by $a' \otimes u \rightarrow a' \otimes \bar{u}$. Let k' be a degree 2 extension of ℓ such that $A' \otimes_\ell k' = M_2(k')$ and let K be the compositum of k, k' : $K = k \otimes_\ell k'$. Then the algebra $A \otimes_k K$ is $A' \otimes_\ell K = M_2(k') \otimes_\ell k = M_2(K)$. The involution of second kind $a \rightarrow \bar{a}$ extends to $M_2(K)$ and acts on each entries fixing $M_2(k') \subset M_2(K)$. The embedding

$$j : A \hookrightarrow M_2(K)$$

is equivariant for the action of the involution: $\forall a \in A, j(\bar{a}) = \overline{j(a)}$, where the action on the left hand side is the conjugation on each entries of the matrix.

The group $j(\Gamma(1))$ is a subgroup of $PSL_2(\mathcal{O}_K)$. Let $I \subset \mathcal{O}_K$ be the (non-trivial) ideal generated by the elements $\bar{a} - a, a \in \mathcal{O}_K$. The ring \mathcal{O}_K/I is a finite ring therefore the subgroup Γ/H of $PSL_2(\mathcal{O}_K/I)$ is finite.

The fundamental group of X/σ is (isomorphic to) Γ/N which is a quotient of the finite group Γ/H , therefore $\pi_1(X/\sigma)$ is finite. \square

4. EXAMPLES

4.1. Aim and terminology. Our goal is to find examples of smooth quaternionic Shimura surfaces X_Γ together with an involution σ on X_Γ having one-dimensional fixed locus. So, we consider an indefinite quaternion algebra A over a totally real field k of degree $n = [k : \mathbb{Q}]$, unramified exactly at two infinite places of k and consider groups Γ commensurable with $\Gamma_{\mathcal{O}}(1)$, the group of norm-1 elements of a maximal order $\mathcal{O} \subset A$ (modulo center).

Definition 18. Let k, A, \mathcal{O} be as above. We say that a discrete group Γ in the commensurability class of $\Gamma_{\mathcal{O}}(1)$ is *admissible of type e* if:

- (1) Γ is torsion-free.
- (2) $e(X_\Gamma) = e$ where $e(X_\Gamma)$ is denoting the (orbifold-) Euler number.
- (3) On A there exists an involution τ of second kind such that Γ is invariant under τ .

Let us remark that according to Proposition 13, the quotient X_Γ/σ will be of general type if $e(X_\Gamma) \leq 36$. Hence, we will focus on admissible groups of type $e = 12, 16, 20, \dots, 36$. Also, the proposition 6 gives us a condition which guarantees the existence of an involution of second kind on A .

4.2. Smoothness and the Euler number. Let $A = A(k, \mathfrak{p}_1, \dots, \mathfrak{p}_{2m})$ be as above and assume that there exists a k/ℓ -involution on A with respect to a subfield $\ell \subset k$. According to Proposition 6 we particularly assume that the primes $\mathfrak{p}_i, i = 1, \dots, 2m$ in \mathcal{O}_k come in pairs: there exist primes $\mathcal{P}_1, \dots, \mathcal{P}_m$ of \mathcal{O}_ℓ such that $\mathfrak{p}_1 \mathfrak{p}_2 = (\mathcal{P}_1), \dots, \mathfrak{p}_{2m-1} \mathfrak{p}_{2m} = (\mathcal{P}_m)$.

If Γ is commensurable with $\Gamma_{\mathcal{O}}(1)$, we have the following general formula for the orbifold Euler number of the Shimura surface X_Γ

Proposition 19. (see [Shi63], [Vig76]) *Let k and $A = A(k, \mathfrak{p}_1, \dots, \mathfrak{p}_{2m})$ be as above. Assume that there exists a k/ℓ -involution on A . Let $n = [k : \mathbb{Q}]$, $\zeta_k(\cdot)$ be the Dedekind zeta function of k and d_k denote the discriminant of k . Then the orbifold Euler number of X_Γ equals*

$$e(X_\Gamma) = [\Gamma_{\mathcal{O}}(1) : \Gamma] \cdot \frac{d_k^{3/2} \zeta_k(2)}{2^{2n-3} \pi^{2n}} \prod_{i=1}^m (N\mathcal{P}_i - 1)^2,$$

where $N\mathcal{P} = |\mathcal{O}_\ell/\mathcal{P}|$ denotes the norm of \mathcal{P} and where

$$[\Gamma_{\mathcal{O}(1)} : \Gamma] = \frac{[\Gamma_{\mathcal{O}(1)} : \Gamma \cap \Gamma_{\mathcal{O}(1)}]}{[\Gamma : \Gamma \cap \Gamma_{\mathcal{O}(1)}]}$$

is the generalized index of two commensurable groups.

When $k = \mathbb{Q}(\sqrt{d})$ is a real quadratic field, we have a particularly handable formula $\zeta_k(2) = \frac{\pi^4 B_{2,k}}{6d_k^{3/2}}$ for the value $\zeta_k(2)$ in terms of the second generalized Bernoulli number. This implies

Corollary 20. (see [Sha78]) *Let $k = \mathbb{Q}(\sqrt{d})$ be a real quadratic field. We denote by $B_{2,k}$ the second generalized Bernoulli number associated with the quadratic Dirichlet character $\chi_k(p) = \left(\frac{d_k}{p}\right)$ of k . Then,*

$$e(X_\Gamma) = [\Gamma_{\mathcal{O}(1)} : \Gamma] \cdot \frac{B_{2,k}}{12} \prod_{i=1}^m (p_i - 1)^2,$$

where p_1, \dots, p_m are rational primes such that $\mathfrak{p}_1 \mathfrak{p}_2 = (p_1), \dots, \mathfrak{p}_{2m-1} \mathfrak{p}_{2m} = (p_m)$.

As next, we would like to discuss the question about the smoothness of X_Γ . Note that X_Γ is smooth if and only if Γ is torsion-free. Here, we will concentrate on subgroups $\Gamma \subset \Gamma_{\mathcal{O}(1)}$. It is worth to recall that the torsions in $\Gamma_{\mathcal{O}(1)}$ correspond to ring embeddings $\mathcal{O}_k[\xi_n] \rightarrow \mathcal{O}$ of the roots of unity into the maximal order \mathcal{O} . The general criterion for the existence of torsions in $\Gamma_{\mathcal{O}(1)}$ is as follows.

Lemma 21. (see [Sha78]) *Let ξ_n be a primitive n -th root of unity. There exists an element of order n in $\Gamma_{\mathcal{O}(1)}$ if and only if:*

- (1) $\xi_n + \xi_n^{-1} \in k$
- (2) every ramified prime in A is non-split in $k(\xi_n)$

The above lemma already gives us a bound for the order of possible torsion elements. Namely, for any $a \in A \setminus k$, the algebra $k(a)$ is commutative subfield of A . Since $\dim_k A = 4$, and $a \notin k$, $\dim_k k(a) = 2$ and $k(a)$ is a quadratic extension of k . Assume now that ξ is a primitive n -th root of unity embedded in \mathcal{O} , then as $L = k(\xi) \subset A$ is a quadratic extension of k , we have $\varphi(n) \leq 2[k : \mathbb{Q}]$, since $\varphi(n) = [\mathbb{Q}(\xi) : \mathbb{Q}] \leq [L : \mathbb{Q}] \leq 2[k : \mathbb{Q}]$.

Above results provide us with criteria to test the conditions in the definition of admissible groups. Let us state a classification result in the case of a real quadratic field k .

Theorem 22. *Let $k = \mathbb{Q}(\sqrt{d})$ be a real quadratic field and consider the totally indefinite quaternion algebra $A = A(k, \mathfrak{p}_1, \overline{\mathfrak{p}}_1, \dots, \mathfrak{p}_m, \overline{\mathfrak{p}}_m)$ over k ramified at the prime ideals dividing rational primes p_1, \dots, p_m which are split in k . If $\Gamma \subset \Gamma_{\mathcal{O}(1)}$ is an admissible group of type e , then the possibilities are as follows:*

type e	k	$\text{Ram}(A)$	$[\Gamma_{\mathcal{O}}(1) : \Gamma]$
12	$\mathbb{Q}(\sqrt{17})$	$\mathfrak{p}_2, \bar{\mathfrak{p}}_2$	18
16	$\mathbb{Q}(\sqrt{13})$	$\mathfrak{p}_3, \bar{\mathfrak{p}}_3$	12
16	$\mathbb{Q}(\sqrt{17})$	$\mathfrak{p}_2, \bar{\mathfrak{p}}_2$	24
20	$\mathbb{Q}(\sqrt{17})$	$\mathfrak{p}_2, \bar{\mathfrak{p}}_2$	30
24	$\mathbb{Q}(\sqrt{13})$	$\mathfrak{p}_3, \bar{\mathfrak{p}}_3$	18
24	$\mathbb{Q}(\sqrt{17})$	$\mathfrak{p}_2, \bar{\mathfrak{p}}_2$	36
24	$\mathbb{Q}(\sqrt{2})$	$\mathfrak{p}_7, \bar{\mathfrak{p}}_7$	4
24	$\mathbb{Q}(\sqrt{33})$	$\mathfrak{p}_2, \bar{\mathfrak{p}}_2$	12
28	$\mathbb{Q}(\sqrt{17})$	$\mathfrak{p}_2, \bar{\mathfrak{p}}_2$	42
32	$\mathbb{Q}(\sqrt{13})$	$\mathfrak{p}_3, \bar{\mathfrak{p}}_3$	24
32	$\mathbb{Q}(\sqrt{17})$	$\mathfrak{p}_2, \bar{\mathfrak{p}}_2$	48
32	$\mathbb{Q}(\sqrt{28})$	$\mathfrak{p}_3, \bar{\mathfrak{p}}_3$	6
36	$\mathbb{Q}(\sqrt{17})$	$\mathfrak{p}_2, \bar{\mathfrak{p}}_2$	54
36	$\mathbb{Q}(\sqrt{33})$	$\mathfrak{p}_2, \bar{\mathfrak{p}}_2$	18

In order to prove this theorem we will need the following elementary lemma.

Lemma 23. *Let G be an arbitrary group and $H \subset G$ a torsion-free subgroup of finite index. If $T \subset G$ is a finite subgroup, then $|T|$ divides $[G : H]$.*

Proof. Let $G/H = \{g_1H, \dots, g_IH\}$ be a set of left cosets of H in G . The group T acts by left multiplication on this set. And moreover this action is free. Otherwise, we would have $t \cdot g_iH = g_iH$ with some non-trivial $t \in T$ and consequently, $g_i^{-1}tg_i \in H$, which is not possible, since H is torsion-free and t as well as $g_i^{-1}tg_i$, is of finite order. By elementary group theory, the length of any T -orbit on G/H is the same as the order of $|T|$. And since G/H is the union of different T -orbits, $|G/H|$ is divisible by $|T|$. \square

Proof of Theorem 22. Recall that we restrict the type e of an admissible group to values $e = 12 + 4k \leq 36$. If $\Gamma \subset \Gamma_{\mathcal{O}}(1)$ is admissible of type e , then

$$36 \geq [\Gamma_{\mathcal{O}}(1) : \Gamma] \frac{B_{2,k}}{12} \prod_{i=1}^m (p_i - 1)^2.$$

Since $[\Gamma_{\mathcal{O}}(1) : \Gamma]$ and $(p_i - 1)^2$ are positive we have the condition $36 \geq \frac{B_{2,k}}{12}$. By [Sha78], Proposition 3.2, the Bernoulli number $B_{2,k}$ is bounded below by $3d_k^{3/2}/50$ and this implies the upper bound $d_k < 373$ for the discriminant of k . Using the formula for the second generalized Bernoulli number given in [Sha78], p. 228, we can compute all the values $B_{2,k}$ for $d_k < 373$ easily with the help of a computer. With the list of all these values we check the necessary conditions:

- $36 \geq B_{2,k}/12$
- $B_{2,k} \mid 12 \cdot e = 144, 192, 240, \dots, 432$ for integral $B_{2,k}$ ($\Leftrightarrow d_k \neq 5$) and with obvious modification for $d_k = 5$.

- The square part of $12e/B_{2,k}$ is divisible by the product $\prod_p (p-1)^2$ where p runs over subsets of rational primes which are split in k (note that $p = 2$, if split in k , contributes the factor 1 to the product).

We obtain the following list of tuples satisfying all the conditions

d_k	$B_{2,k}$	e	$12e/B_{2,k} = I \cdot \prod (p-1)^2$
137	192	16	$1 = (2-1)^2$
113	144	12	$1 = (2-1)^2$
109	108	36	$4 = (3-1)^2$
105	144	12	$1 = (2-1)^2$
85	72	24	$1 = (3-1)^2$
40	28	28	$12 = 3 \cdot (3-1)^2$
37	20	20	$12 = 3 \cdot (3-1)^2$
33	24	24	$12 \cdot (2-1)^2$
29	12	16	$16 = (5-1)^2$
29	12	32	$32 = 2 \cdot (5-1)^2$
29	12	36	$36 = (7-1)^2$
28	16	16	$12 = 3 \cdot (3-1)^2$
28	16	32	$24 = 6(3-1)^2$
24	12	16	$16 = (5-1)^2$
24	12	32	$32 = 2 \cdot (5-1)^2$
17	8	$12 + 4k, k = 0, \dots, 6$	$e \cdot (2-1)^2$
13	4	$12 + 4k, k = 0, \dots, 6$	$(9 + 3k) \cdot (3-1)^2$
8	2	12, 24, 36	$2 \cdot (7-1)^2, 4 \cdot (7-1)^2, 6 \cdot (7-1)^2$
5	$\frac{4}{5}$	20	$3 \cdot (11-1)^2$

We observe (keeping also in mind the splitting behavior of 2 in k) that the set of ramified places in A is determined by the value $12e/B_{2,k}$ in the table.

As next we identify those subgroups $\Gamma \subset \Gamma_{\mathcal{O}}(1)$ which cannot be torsion-free. For this we use the two Lemmas 21 and 23. Namely, note first that $\Gamma_{\mathcal{O}}(1)$ contains at most torsions of order 2, 3 and 6 for $k \neq \mathbb{Q}(\sqrt{5}), \mathbb{Q}(\sqrt{2})$ and additionally elements of order 5 for $k = \mathbb{Q}(\sqrt{5})$ and of order 4 for $k = \mathbb{Q}(\sqrt{2})$. Case by case analysis leads to the final statement; to check the splitting behavior in $k(\xi_n)$ one can use the criterion of Shavel (see [Sha78], Theorem 4.8). We double-checked the the conditions of Lemma 21 explicitly with PARI/GP.

4.3. Admissible groups defined by congruences. Let k a totally real number field and $A(k, \mathfrak{p}_1, \bar{\mathfrak{p}}_1, \dots, \mathfrak{p}_m, \bar{\mathfrak{p}}_m)$ an indefinite quaternion algebra over k , \mathcal{O} a maximal order in A , \mathcal{O} and $\Gamma_{\mathcal{O}}(1)$ as in the previous sections. If \mathfrak{a} is a two-sided \mathcal{O} ideal in \mathcal{O} , the *principal congruence subgroup* in $\mathcal{O}(1)$ associated with \mathfrak{a} is defined as

$$\mathcal{O}(\mathfrak{a}) = \{g \in \mathcal{O} \mid \text{Nrd}(g) = 1, g - 1 \in \mathfrak{a}\}$$

Additionally we define $\Gamma_{\mathcal{O}}(\mathfrak{a}) = \mathcal{O}(\mathfrak{a})/Z$ where Z denotes the center of $\mathcal{O}(\mathfrak{a})$. A *congruence subgroup* in $\mathcal{O}(1)$, resp. $\Gamma_{\mathcal{O}}(1)$, is a subgroup $G \subset \mathcal{O}(1)$,

resp. $\Gamma \subset \Gamma_{\mathcal{O}}(1)$, which contains some $\mathcal{O}(\mathfrak{a})$, resp. $\Gamma_{\mathcal{O}}(\mathfrak{a})$. The group $\mathcal{O}(\mathfrak{a})$ is a normal subgroup of finite index in $\mathcal{O}(1)$ and we have $\mathcal{O}(1)/\mathcal{O}(\mathfrak{a}) \cong \Gamma_{\mathcal{O}}(1)/\pm \Gamma_{\mathcal{O}}(\mathfrak{a})$ if $2 \notin \mathfrak{a}$ and $\mathcal{O}(1)/\mathcal{O}(\mathfrak{a}) \cong \Gamma_{\mathcal{O}}(1)/\Gamma_{\mathcal{O}}(\mathfrak{a})$ if $2 \in \mathfrak{a}$. The size $|\mathcal{O}(1)/\mathcal{O}(\mathfrak{a})|$ is computed as follows

- Any two-sided ideal \mathfrak{a} in \mathcal{O} has a unique decomposition $\mathfrak{a} = \mathfrak{Q}_1^{e_1} \cdots \mathfrak{Q}_r^{e_r}$ as a product of prime ideal powers. Then, $\mathcal{O}(1)/\mathcal{O}(\mathfrak{a})$ is a direct product

$$\mathcal{O}(1)/\mathcal{O}(\mathfrak{a}) = \mathcal{O}(1)/\mathcal{O}(\mathfrak{Q}_1^{e_1}) \times \cdots \times \mathcal{O}(1)/\mathcal{O}(\mathfrak{Q}_r^{e_r}).$$

- Let \mathfrak{Q} be a prime ideal in \mathcal{O} . The \mathcal{O}_k -ideal $\mathfrak{q} = \text{Nrd}(\mathfrak{Q})$ generated by the reduced norms of elements in \mathfrak{Q} , which is also the intersection $\mathfrak{q} = \mathfrak{Q} \cap \mathcal{O}_k$, is a prime ideal and there are two possible cases:
 - $\mathfrak{q} \notin \text{Ram}(A)$. Then, $\mathfrak{Q} = \mathfrak{q}\mathcal{O}$ and $\mathcal{O}(1)/\mathcal{O}(\mathfrak{Q}_1^e) \cong \text{SL}_2(\mathcal{O}_k/\mathfrak{q}^e)$.
 - $\mathfrak{q} \in \text{Ram}(A)$. Then, $\mathfrak{Q}^2 = \mathfrak{q}\mathcal{O}$ and $\mathcal{O}(1)/\mathcal{O}(\mathfrak{Q}^e) \cong (\mathcal{O}/\mathfrak{Q}^e)_1 = \ker \left((\mathcal{O}/\mathfrak{Q}^e)^* \xrightarrow{Nr} (\mathcal{O}_k/\mathfrak{q}^e)^* \right)$, where Nr is the norm map induced by the reduced norm $\text{Nrd} : \mathcal{O} \rightarrow \mathcal{O}_k$.

Remark 24. If we want to search for admissible groups among the principal congruence subgroups, then we must note the following: for $g \in \mathcal{O}(\mathfrak{Q})$ we have $\bar{g} \in \mathcal{O}(\bar{\mathfrak{Q}})$, so, $\Gamma_{\mathcal{O}}(\mathfrak{Q})$ will be admissible if and only if $\mathfrak{Q} = \bar{\mathfrak{Q}}$ (for more precise statement see Theorem 26 and Lemma 27 below). This is already a strong condition: If k is a quadratic field, the prime \mathfrak{q} under \mathfrak{Q} must be inert or ramified over ℓ . For instance, this in combination with list of possible candidates from Theorem 22 shows there are no admissible principal congruence subgroups of any type e defined over a real quadratic field.

In the following we will make use of the following well-known fact.

Lemma 25. *Let \mathfrak{q} be a prime ideal in \mathcal{O}_k , unramified in a k -central quaternion algebra A and $\mathfrak{Q} = \mathfrak{q}\mathcal{O}$ the corresponding \mathcal{O} -ideal. Let $q\mathbb{Z} = \mathfrak{q} \cap \mathbb{Z}$ be the rational prime divisible by \mathfrak{q} and finally, let $x \in \mathcal{O}^1(\mathfrak{Q})$ be an element of order p , where p is a prime. Then $p = q$.*

Proof. We have $x \in \mathcal{O}^1(\mathfrak{Q}) \Leftrightarrow \text{Nrd}(x - 1) \in \mathfrak{q}$. We can assume that x is a primitive p -th root of unity contained in A . Since $k(x) \subset A$, we have $\text{Nrd}(x - 1) = N_{k(x)/k}(x - 1)$. Taking $N_{k/\mathbb{Q}}(\cdot)$ on both sides we obtain

$$N_{k/\mathbb{Q}}(\text{Nrd}(x - 1)) = N_{k(x)/\mathbb{Q}}(x - 1) \in N_{k/\mathbb{Q}}(\mathfrak{q}) = q^f \mathbb{Z}.$$

where $f > 0$ is the inertia degree of \mathfrak{q} . On the other hand $x - 1 \in \mathbb{Q}(x)$, and therefore $N_{k(x)/\mathbb{Q}}(x - 1) = N_{\mathbb{Q}(x)/\mathbb{Q}}(x - 1)^d$, where $d = [k(x) : \mathbb{Q}(x)]$. Altogether, we obtain the relation

$$N_{\mathbb{Q}(x)/\mathbb{Q}}(x - 1)^d \in q^f \mathbb{Z}.$$

Finally, it is well-known that $N_{\mathbb{Q}(x)/\mathbb{Q}}(x - 1) = \pm p$ and from this the claim follows. \square

Assume that the prime ideal $\mathfrak{q} \subset \mathcal{O}_k$ is unramified in A , then $\mathfrak{q}\mathcal{O}$ is a prime ideal in $\mathcal{O} \subset A$. Since in this case $\mathfrak{Q} = \mathfrak{q}\mathcal{O}$ we will write $\Gamma_{\mathcal{O}}(\mathfrak{q}) = \Gamma_{\mathcal{O}}(\mathfrak{Q})$. Let $s = q^f$ be the absolute norm $N_{k/\mathbb{Q}}(\mathfrak{q}) = |\mathcal{O}_k/\mathfrak{q}|$ of \mathfrak{q} . Then $\Gamma_{\mathcal{O}}(1)/\Gamma_{\mathcal{O}}(\mathfrak{q}) \cong PSL_2(\mathcal{O}_k/\mathfrak{q}) \cong PSL_2(s) = PSL_2(\mathbb{F}_s)$. By the classification theorem of Dickson, we know all the subgroups of $PSL_2(s)$. Let us mention two particular subgroups which we will use later on:

- (1) Borel subgroup $B \subset PSL_2(s)$ consisting of all upper triangular matrices in $PSL_2(s)$. The group B is a maximal subgroup of $PSL_2(s)$ of index $s+1$ and order $s(s-1)/t$, with $t = \gcd(s-1, 2)$.
- (2) Unipotent subgroup $U \subset PSL_2(s)$ consisting of all elements in B of the form $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$. The group U is a subgroup of index $(s^2-1)/t$ and order s .

With above notations let $\pi : \Gamma_{\mathcal{O}}(1) \rightarrow PSL_2(s)$ denote the epimorphism induced by the canonical projection $\Gamma_{\mathcal{O}}(1) \rightarrow Q = \Gamma_{\mathcal{O}}(1)/\Gamma_{\mathcal{O}}(\mathfrak{q})$. Let $\Gamma^B(\mathfrak{q}) = \pi^{-1}(B)$ and $\Gamma^U(\mathfrak{q}) = \pi^{-1}(U)$ be the inverse images of B and U respectively. These are subgroups of $\Gamma_{\mathcal{O}}(1)$ of index equal to the index of its image in $PSL_2(s)$ under π . It is important to mention that $\Gamma_{\mathcal{O}}^B(\mathfrak{q})$ is also constructed as the group of the norm-1 elements (modulo center) in an appropriate Eichler order \mathcal{E} . The construction is as follows: Let \mathcal{O} be a maximal order and denote $\mathcal{O}_v = \mathcal{O} \otimes_{\mathcal{O}_k} R_v$ the localizations of \mathcal{O} at finite places v of k . There R_v denotes the valuation ring in the localization k_v of k at v . Note that $\mathcal{O} = \bigcap_v \mathcal{O}_v$. Let \mathcal{O}' be another maximal order with the property that for all finite places $v \neq \mathfrak{q}$ we have $\mathcal{O}_v = \mathcal{O}'_v$ and additionally $\mathcal{O}_{\mathfrak{q}} \cap \mathcal{O}'_{\mathfrak{q}}$ has index $N(\mathfrak{q}) = \#R_{\mathfrak{q}}/\mathfrak{q}R_{\mathfrak{q}}$ in both, $\mathcal{O}_{\mathfrak{q}}$ and $\mathcal{O}'_{\mathfrak{q}}$. Put $\mathcal{E} = \mathcal{O} \cap \mathcal{O}'$. Then, by definition, \mathcal{E} is an Eichler order of level \mathfrak{q} . If \mathfrak{q} is ramified in A , we have $\mathcal{E} = \mathcal{O}$ and in the case where \mathfrak{q} is unramified, after a possible conjugation, we can assume that $\mathcal{O}_{\mathfrak{q}} = M_2(R_{\mathfrak{q}})$ and we can choose $\mathcal{O}'_{\mathfrak{q}} = PM_2(R_{\mathfrak{q}})P^{-1}$ with $P = \text{diag}(1, \varpi)$, where ϖ is a generator of the valuation ideal $\mathfrak{q}R_{\mathfrak{q}}$. The reduction modulo \mathfrak{q} maps $\mathcal{E}_{\mathfrak{q}} = \mathcal{O}_{\mathfrak{q}} \cap \mathcal{O}'_{\mathfrak{q}}$ surjectively to the subalgebra of upper triangular matrices in $M_2(R_{\mathfrak{q}}/\mathfrak{q}R_{\mathfrak{q}})$. Therefore the group \mathcal{E}^1 of norm-1 elements in \mathcal{E} corresponds to exactly those elements in \mathcal{O}^1 which modulo $\mathfrak{q}\mathcal{O}$ are upper triangular.

In general, a k/ℓ -involution on a quaternion algebra A does not preserve a maximal order. But under certain conditions on A we can ensure the existence of such an order:

Theorem 26. (*Scharlau, [Sch84, Theorem 4.6]*) *Let A be a quaternion algebra over k admitting a k/ℓ -involution τ . Then there exists a maximal order \mathcal{O} invariant under τ unless the following exceptional situation is given:*

- the extension k/ℓ is unramified and
- the number of places $v \in \text{Ram}(A)$ is $\equiv 2 \pmod{4}$.

Corollary 27. *Assume that A admits a k/ℓ -involution τ which preserves a maximal order \mathcal{O} and let $\mathfrak{q} \subset \mathcal{O}_k$ be a prime ideal which is unramified in A and $\mathfrak{q}\mathcal{O}$ the corresponding prime ideal in \mathcal{O} . Assume that the non-trivial*

k/ℓ -automorphism $x \mapsto \bar{x}$ maps \mathfrak{q} to itself, that is, $\bar{\mathfrak{q}} = \mathfrak{q}$. Then, for each of the groups $\Gamma = \Gamma_{\mathcal{O}}(1)$, $\Gamma_{\mathcal{O}}^B(\mathfrak{q})$, $\Gamma_{\mathcal{O}}^U(\mathfrak{q})$ and $\Gamma_{\mathcal{O}}(\mathfrak{q})$ there exists a k/ℓ -involution on A leaving Γ invariant.

Proof. The group $\Gamma_{\mathcal{O}}(1)$ is τ -invariant since \mathcal{O} is τ -invariant and for any $a \in A$ we have $Nrd(\bar{a}) = \overline{Nrd(a)}$. Also, $\Gamma(\mathfrak{q})$ is invariant, since for every $x \in \Gamma_{\mathcal{O}}(\mathfrak{q})$ there is a representative $x' \in \mathcal{O}$ of the class x satisfying $x-1 \in \mathfrak{q}\mathcal{O}$, thus $(x'-1) \in \bar{\mathfrak{q}}\mathcal{O} = \bar{\mathfrak{q}}\mathcal{O} = \mathfrak{q}\mathcal{O}$, by assumption, hence $\bar{x}' \equiv 1 \pmod{\mathfrak{q}\mathcal{O}}$.

In order to prove the invariance of other groups, we first localize at \mathfrak{q} ; since $\mathfrak{q} \notin \text{Ram}(A)$, the local algebra $A_{\mathfrak{q}} = A \otimes_k k_{\mathfrak{q}}$ is isomorphic to $M_2(k_{\mathfrak{q}})$ and since \mathcal{O} is maximal $\mathcal{O}_{\mathfrak{q}} = \mathcal{O} \otimes_{\mathcal{O}_k} R_{\mathfrak{q}}$ is isomorphic to $M_2(R_{\mathfrak{q}})$ where $R_{\mathfrak{q}}$ is the valuation ring in $k_{\mathfrak{q}}$. Choosing the appropriate isomorphism $A_{\mathfrak{q}} \cong M_2(k_{\mathfrak{q}})$ we can assume that $\mathcal{O}_{\mathfrak{q}} = M_2(R_{\mathfrak{q}})$. Consider the order $\mathcal{E}_{\mathfrak{q}} = M_2(R_{\mathfrak{q}}) \cap PM_2(R_{\mathfrak{q}})P^{-1}$ with $P = \text{diag}(1, \varpi)$, where ϖ is a generator of the valuation ideal $\mathfrak{q}R_{\mathfrak{q}}$. This is the localization of the global Eichler order \mathcal{E} corresponding to a group $\Gamma_{\mathcal{O}}^B(\mathfrak{q})$. The involution τ which leaves \mathcal{O} invariant extends to an involution $\hat{\tau}$ on $\mathcal{O}_{\mathfrak{q}} = \mathcal{O} \otimes_{\mathcal{O}_k} R_{\mathfrak{q}}$ in an obvious way by defining $\hat{\tau}(x \otimes r) = \tau(x) \otimes \bar{r}$ where $r \mapsto \bar{r}$ is the generator of $\text{Gal}(k_{\mathfrak{q}}/\ell_{\mathcal{Q}})$ and $\mathcal{Q} = \mathfrak{q} \cap \ell$ is the prime of ℓ lying under \mathfrak{q} and $\ell_{\mathcal{Q}}$ its localization. By this the involution $\hat{\tau}$ maps the matrix P to $\pm P$ depending on whether $k_{\mathfrak{q}}/\ell_{\mathcal{Q}}$ is unramified or ramified but in any case $\hat{\tau}$ preserves $\mathcal{E}_{\mathfrak{q}}$. From the construction of $\hat{\tau}$ we see that $\hat{\tau}$ also preserves $\mathcal{O} \cap \mathcal{E}_{\mathfrak{q}} = \mathcal{E}$ and particularly the norm-1 group \mathcal{E}^1 whose quotient by the center is $\Gamma_{\mathcal{O}}^B(\mathfrak{q})$. The group $\Gamma_{\mathcal{O}}^U(\mathfrak{q})$ consists of those elements in $\Gamma_{\mathcal{O}}^B(\mathfrak{q})$ which reduce modulo \mathfrak{q} to upper triangular matrices with only 1 on the diagonal. The preimage of such matrices in $\mathcal{E}_{\mathfrak{q}}$ is preserved by $\hat{\tau}$ and hence τ preserves the preimage of these matrices in $\mathcal{O} \cap \mathcal{E}_{\mathfrak{q}}$. This implies that with $\Gamma_{\mathcal{O}}^B(\mathfrak{q})$ also $\Gamma_{\mathcal{O}}^U(\mathfrak{q})$ is preserved. \square

4.4. Construction with the Borel subgroup. Let $A(k, \mathfrak{p}_1, \bar{\mathfrak{p}}_1, \dots, \mathfrak{p}_m, \bar{\mathfrak{p}}_m)$ be as before. Let \mathfrak{q} be a prime ideal of k such that $\mathfrak{q} \neq \mathfrak{p}_i, \bar{\mathfrak{p}}_i$ for $i = 1, \dots, m$ and consider the group $\Gamma_{\mathcal{O}}^B(\mathfrak{q})$, the inverse image $\pi^{-1}(B)$ of a Borel subgroup $B \subset \Gamma_{\mathcal{O}}(1)/\Gamma_{\mathcal{O}}(\mathfrak{q}) \cong PSL_2(\mathcal{O}_k/\mathfrak{q})$. The group $\Gamma_{\mathcal{O}}^B(\mathfrak{q})$ is a subgroup of index $N_{k/\mathbb{Q}}(\mathfrak{q}) + 1$ in $\Gamma_{\mathcal{O}}(1)$. In order to discuss the torsions in $\Gamma_{\mathcal{O}}^B(\mathfrak{q})$ it will again be useful to interpret $\Gamma_{\mathcal{O}}^B(\mathfrak{q})$ as the norm-1 group of an Eichler order as explained in previous section. Let us give conditions under which $\Gamma_{\mathcal{O}}^B(\mathfrak{q})$ is torsion-free.

Lemma 28. *Let k, A, \mathcal{O} and \mathfrak{q} be as above. Then, $\Gamma_{\mathcal{O}}^B(\mathfrak{q})$ contains a torsion if and only if a primitive n -th root of unity ξ can be embedded in $\mathcal{E}(\mathfrak{q})$. This happens if and only if every prime $\mathfrak{p} \in \text{Ram}(A)$ is either ramified or inert in $k(\xi)$ and \mathfrak{q} is split in $k(\xi)$.*

Proof. Let γ be a torsion in $\Gamma_{\mathcal{O}}^B(\mathfrak{q})$. Then there is a minimal N such that $\gamma^N = \pm 1$, which implies that γ is an N -th (or $2N$ -th) root of unity contained in $\mathcal{E}(\mathfrak{q})$. Conversely, let ξ be a root of unity, let $L = k(\xi)$ and assume that there exists an embedding $\sigma : L \hookrightarrow A$ such that $\sigma(L) \cap \mathcal{E}(\mathfrak{q}) = \mathcal{O}_k(\xi)$ (that is,

$\mathcal{O}_k(\xi)$ is embedded in $\mathcal{E}(\mathfrak{q})$). Then $\sigma(\xi)$ is a torsion in $\Gamma_{\mathcal{O}}^B(\mathfrak{q})$. By a theorem of Eichler (see [Eic55, Satz 6]) such an embedding is possible if and only if the splitting condition mentioned in the statement of Lemma is satisfied. \square

4.5. A Shimura surface with an involution of second kind and $p_g = 5$. Let $k = \mathbb{Q}(\sqrt{33})$ and $A = A(k, \mathfrak{p}_2 \bar{\mathfrak{p}}_2)$ the indefinite quaternion algebra ramified exactly at the two places over 2 (note that 2 is split in k , since $33 \equiv 1 \pmod{8}$). By Theorem 6, A admits a k/\mathbb{Q} -involution and since k/\mathbb{Q} is not totally ramified, Theorem 26 ensures the existence of an involution invariant order \mathcal{O} . Let $\mathfrak{q} = \mathfrak{q}_{11}$ be the prime over 11. Since 11 is ramified in k , we have $N_{k/\mathbb{Q}}(\mathfrak{q}) = 11$ and $\Gamma_{\mathcal{O}}^B(\mathfrak{q}_{11})$ is of index 12 in $\Gamma_{\mathcal{O}}$. By the volume formula from Theorem 19, we have $e(\Gamma_{\mathcal{O}}(1)) = 2$, hence $e(\Gamma_{\mathcal{O}}^B(\mathfrak{q})) = 24$. Let us show that $\Gamma_{\mathcal{O}}^B(\mathfrak{q})$ is torsion-free. For this we need to exclude the existence of elements of order 2 and 3 only, since these are the only primes for which an embedding of ξ_p in A is possible. Elements of order 2 come from embeddings of $k(\xi_4) = k(\sqrt{-1})$ in A and those of order 3 from embeddings of $k(\xi_6) = k(\sqrt{-3})$. We use Lemma 28: $k(\xi_4) \cong \mathbb{Q}[x]/\langle x^4 - 64x^2 + 1156 \rangle$ and we find that $11\mathcal{O}_{k(\xi_4)} = \mathfrak{Q}^2$ with $(\mathcal{O}_{k(\xi_4)}/\mathfrak{Q} : \mathbb{F}_{11}) = 2$. It follows that \mathfrak{q}_{11} is inert in $k(\xi_4)$ and by Lemma 28, $\Gamma_{\mathcal{O}}^B(\mathfrak{q})$ contains no elements of order 2. Similar argument excludes the existence of elements of order 3. Namely, $k(\xi_6) \cong \mathbb{Q}[x]/\langle x^4 - 60x^2 + 1296 \rangle$ and in $k(\xi_6)$ we again have $11\mathcal{O}_{k(\xi_6)} = \mathfrak{Q}^2$ with a prime ideal \mathfrak{Q} in $\mathcal{O}_{k(\xi_6)}$ whose inertia degree is $(\mathcal{O}_{k(\xi_6)}/\mathfrak{Q} : \mathbb{F}_{11}) = 2$. Again this implies that \mathfrak{q}_{11} is inert in $\mathcal{O}_{k(\xi_6)}$ and by Lemma 28, there are no elements of order 3 in $\Gamma_{\mathcal{M}}^B(\mathfrak{q}_{11})$. Finally by Corollary 27, $\Gamma_{\mathcal{O}}^B(\mathfrak{q})$ is invariant under the involution on \mathcal{O} and we get:

Theorem 29. *The group $\Gamma_{\mathcal{O}}^B(\mathfrak{q}_{11})$ is admissible of type 24.*

Remark 30. Unfortunately, one promising candidate for a admissible group of type 12 ($p_g = 2$) fails to be torsion-free. Namely, let $A = A(\mathbb{Q}(\sqrt{17}), \mathfrak{p}_2 \bar{\mathfrak{p}}_2)$ and take $\mathfrak{q} = \mathfrak{q}_{17}$, the ideal over 17. Then the index of $\Gamma_{\mathcal{O}}^B(\mathfrak{p}_{17})$ in $\Gamma_{\mathcal{O}}(1)$ is 18 and as $e(\Gamma_{\mathcal{O}}(1)) = 2/3$, we get $e(\Gamma_{\mathcal{O}}^B(\mathfrak{p}_{17})) = 12$. The invariance under the involution of second kind is guaranteed by the condition $\mathfrak{q}_{17} = \bar{\mathfrak{q}}_{17}$. But in $L = k(\xi_4) \cong \mathbb{Q}[x]/\langle x^4 - 32x^2 + 324 \rangle$, both primes \mathfrak{p}_2 and $\bar{\mathfrak{p}}_2$ are non-split and \mathfrak{p}_{17} is split. This implies that there are 2-torsions in $\Gamma_{\mathcal{O}}^B(\mathfrak{q}_{17})$. There are more examples of non-smooth Shimura surfaces with “good” invariants. Consider for instance $k = \mathbb{Q}(\sqrt{28})$ and $A = A(k, \mathfrak{p}_3 \bar{\mathfrak{p}}_3)$ the indefinite quaternion algebra ramified at the two places over 3. Theorems 6 and 27 ensure that A has an involution of second kind and that there is an order \mathcal{O} invariant under the involution. From Theorem 19 we know that $e(\Gamma_{\mathcal{O}}(1)) = 16/3$. The rational prime 2 is ramified in k . Let $\mathfrak{q} = \mathfrak{q}_2$ be the prime ideal of k with $\mathfrak{q}_2^2 = 2\mathcal{O}_k$ and consider the principal congruence subgroup $\Gamma_{\mathcal{O}}(\mathfrak{q})$. It is a subgroup in $\Gamma_{\mathcal{O}}(1)$ of index 6. There are no elements of order 3 in $\Gamma_{\mathcal{O}}(\mathfrak{q})$ by Lemma 25 but there are elements of order 2 coming from embeddings of $\sqrt{-1}$ in \mathcal{O} .

Looking at the list in Theorem 22, we can also prove that no other admissible groups of type $e = 12, \dots, 36$ can be obtained from $k = \mathbb{Q}(\sqrt{d})$ and $\Gamma = \Gamma_{\mathcal{O}}^B(\mathfrak{q})$ or $\Gamma_{\mathcal{O}}^U(\mathfrak{q})$.

Instead we can consider totally real fields of higher degree:

4.6. A Shimura surface with an involution of second kind and $p_g = 6$.
In this example we consider the unique totally real number field k of degree 4 and discriminant $d_k = 725$. Its defining polynomial is $x^4 - x^3 - 3x^2 + x + 1$ and k contains $\ell = \mathbb{Q}(\sqrt{5})$ as a subfield of degree 2. Let us consider the k -central quaternion algebra $A(k, \emptyset)$ ramified exactly at two infinite places v_1 and v_2 of k such that $v_2 = v_1 \circ \sigma$, where $\langle \sigma \rangle = \text{Gal}(k/\ell)$. We remark that k is not a Galois extension of \mathbb{Q} . The algebra A admits an involution of second kind τ and by Theorem 26, there is a maximal order \mathcal{O} invariant under τ . Consider now the prime $q = 29$. In $\ell = \mathbb{Q}(\sqrt{5})$, $29\mathcal{O}_{\ell} = \mathcal{Q}_{29}\mathcal{Q}'_{29}$ is a product of two primes. On the other hand, a computation with PARI/GP shows that the ideal $29\mathcal{O}_k = \mathfrak{q}_{29}^2\mathfrak{q}'_{29}$ is also a product of two prime ideals \mathfrak{q}_{29} (with multiplicity 2) and \mathfrak{q}'_{29} , hence neither \mathcal{Q}_{29} nor \mathcal{Q}'_{29} is split in k . Moreover we deduce that $\mathfrak{q}_{29}^2 = \mathcal{Q}_{29}\mathcal{O}_k$ and $\mathfrak{q}'_{29} = \mathcal{Q}'_{29}\mathcal{O}_k$ as well as $\mathcal{O}_k/\mathfrak{q}_{29} \cong \mathbb{F}_{29}$ and $\mathcal{O}_k/\mathfrak{q}'_{29} \cong \mathbb{F}_{29^2}$. By Theorem 19 we have $e(X_{\Gamma_{\mathcal{O}}(1)}) = 1/15$ (we compute $\zeta_k(2)$ with PARI/GP command `zetak`). Consider the congruence subgroup $\Gamma_{\mathcal{O}}^U(\mathfrak{q}_{29})$. We obtain $[\Gamma_{\mathcal{O}}(1) : \Gamma_{\mathcal{O}}^U(\mathfrak{q}_{29})] = 420$ and $c_2(X_{\Gamma_{\mathcal{O}}^U(\mathfrak{q}_{29})}) = 28$. By corollary 27, $\Gamma_{\mathcal{O}}^U(\mathfrak{q}_{29})$ is stable under τ . Also $\Gamma_{\mathcal{O}}^U(\mathfrak{q}_{29})$ is torsion-free. Namely, as the order of U is s , any non-trivial torsion element in $\Gamma_{\mathcal{O}}^U(\mathfrak{q}_{29})$ has order 29 (which is impossible by lemma 21) or lies already in $\Gamma_{\mathcal{O}}(\mathfrak{q}_{29})$. But this latter group is torsion-free by Lemma 25.

Remark 31. Of course, the strategy of Section 4.2 leading to Theorem 22 could be applied also in the case of quaternion algebras over totally real fields k of degree > 2 but becomes very soon computationally involved. Restricting ourselves to totally real quartic fields of discriminant $\leq 10^4$ and groups of type $\Gamma_{\mathcal{O}}^B(\mathfrak{q})$ or $\Gamma_{\mathcal{O}}^U(\mathfrak{q})$ we find that example 4.6 is the only example of an admissible group (of any type $e = 12 + 4k$, $0 \leq k \leq 6$). But similarly to Remark 30 we find some interesting non-smooth examples: Let $k_{4,D}$ denote a totally real field of degree 4 and discriminant D (in the examples below, there will be only one such field up to isomorphism) containing a real quadratic field ℓ . Let $A(k_{4,D}, \emptyset)$ denote the quaternion algebra over $k_{4,D}$ ramified exactly at the two infinite places of $k_{4,D}$ which are conjugate under the non-trivial ℓ -automorphism of $k_{4,D}$. Such quaternion algebra admits a $k_{4,D}/\ell$ -involution. Let \mathcal{O} be a maximal order in $A(k_{4,D}, \emptyset)$ and $\Gamma_{\mathcal{O}}(1)$ the corresponding projectivized modular group. We get several *singular* Shimura surfaces X_{Γ} admitting an involution of second kind with $\Gamma \subset \Gamma_{\mathcal{O}}(1)$ given in the table below:

$k_{n,D}$	ℓ	$e(\Gamma(1))$	Γ	$e(\Gamma)$
$k_{4,2624}$	$\mathbb{Q}(\sqrt{5})$	$1/2$	$\Gamma^B(\mathfrak{q}_{7,1}) \cap \Gamma^B(\mathfrak{q}_{7,2})$	32
$k_{4,2000}$	$\mathbb{Q}(\sqrt{5})$	$1/3$	$\Gamma(\mathfrak{q}_5)$	20
$k_{4,2000}$	$\mathbb{Q}(\sqrt{5})$	$1/3$	$\Gamma(\mathfrak{q}_2)$	20
$k_{4,2525}$	$\mathbb{Q}(\sqrt{5})$	$2/3$	$\Gamma^B(\mathfrak{q}_5) \cap \Gamma^B(\overline{\mathfrak{q}}_5)$	24
$k_{4,3600}$	$\mathbb{Q}(\sqrt{3})$	$4/5$	$\Gamma^U(\mathfrak{q}_2)$	12

5. DETERMINATION OF THE FIXED CURVE.

Let $X_\Gamma = \mathbb{H}_{\mathbb{C}}^2 / \Gamma$ be a smooth Shimura surface such that the involution μ on $\mathbb{H}_{\mathbb{C}}^2$ exchanging the factors descends to an involution σ on the quotient X_Γ . The image of the diagonal $\Delta \subset \mathbb{H} \times \mathbb{H}$ is a smooth Shimura curve C_Γ fixed by σ . The aim of this section is to determine that curve in examples we investigated in Sections 4.5 and 4.6.

The analogous problem for Hilbert modular surfaces is well-known, see for instance [Hir73] or [Hau82]. The quaternion algebra A over k defining the group Γ has a non trivial involution τ of second kind. That involution leaves invariant a subfield ℓ . Recall that $A = A(k, \mathfrak{p}_1, \dots, \mathfrak{p}_{2r})$ with $\mathfrak{p}_{2i-1} = \mathfrak{p}_{2i}^\alpha$ as in Prop. 6 and $\alpha : k \rightarrow k$ the involution of the extension k/ℓ with $\sigma_1 = \sigma_2 \circ \alpha$ for $\sigma_i : k \rightarrow \mathbb{R}$ the unramified infinite places in A . Also note that the involution $\sigma : X_\Gamma \rightarrow X_\Gamma$ is determined by the involution of second kind τ on A . The fixed point set of σ is associated with the invariant ℓ -subalgebra $A' = \{a \in A \mid \tau(a) = a\}$ of A . From the proof of Proposition 6 we know that A' is a quaternion algebra over ℓ with the property $A' \otimes k \cong A$. Moreover, A' is ramified at every prime \mathcal{P}_i of ℓ such that $\mathcal{P}_i \mathcal{O}_k = \mathfrak{p}_{2i-1} \mathfrak{p}_{2i}$ for $i = 1, \dots, r$.

Lemma 32. *Let $A = A(k, \mathfrak{p}_1, \dots, \mathfrak{p}_{2r})$ be a quaternion algebra admitting an involution of second kind τ and A' the elementwise τ -invariant subalgebra. Let \mathcal{O} be an order in A , then $\mathcal{O}' = \mathcal{O} \cap A'$ is an order in A' . Conversely, assume that $A' = A'(\ell, \mathcal{P}_1, \dots, \mathcal{P}_s)$ is a quaternion algebra over ℓ and \mathcal{O}' an order in A' then $\mathcal{O} = \mathcal{O}' \otimes_{\mathcal{O}_\ell} \mathcal{O}_k$ is an order in A . Assume that \mathcal{O}' is maximal and each \mathcal{P}_i is split in k then $\mathcal{O}' \otimes_{\mathcal{O}_\ell} \mathcal{O}_k$ is a maximal order in $A = A' \otimes_\ell k$.*

Proof. The first part of the Lemma concerning the correspondence between orders \mathcal{O}' in A' and orders in A is obvious and we shall therefore prove only the second part. Assume that $\mathcal{O}' \subset A'$ is a maximal order and let $\mathcal{O} = \mathcal{O}' \otimes_{\mathcal{O}_\ell} \mathcal{O}_k$. The order \mathcal{O} is maximal if and only if each of its localizations $\mathcal{O}_{\mathfrak{p}}$ is maximal in the local algebra $A_{\mathfrak{p}}$. Here, $\mathcal{O}_{\mathfrak{p}} = \mathcal{O}'_{\mathcal{P}} \otimes \mathcal{O}_{k_{\mathfrak{p}}}$ arises from the local maximal order $\mathcal{O}'_{\mathcal{P}}$ corresponding to a finite place \mathcal{P} of ℓ lying under \mathfrak{p} . Assume now that $\mathcal{P} \neq \mathcal{P}_i$ is a finite place at which B is unramified. Then $\mathcal{O}'_{\mathcal{P}} \cong M_2(\mathcal{O}_{\ell_{\mathcal{P}}})$ and clearly $\mathcal{O}_{\mathfrak{p}} \cong M_2(\mathcal{O}_{k_{\mathfrak{p}}})$ is also maximal. If $\mathcal{P} = \mathcal{P}_i$ is a place such that $A'_{\mathcal{P}_i}$ is a division algebra, as by assumption $k_{\mathfrak{p}_i} = k_{\overline{\mathfrak{p}}_i} = \ell_{\mathcal{P}_i}$, $\mathcal{O}_{\mathfrak{p}_i}$ and $\mathcal{O}_{\overline{\mathfrak{p}}_i}$ are maximal. \square

Remark 33. We shall note that in the case where $A' = A'(\ell, \mathcal{P}_1, \dots, \mathcal{P}_1, \mathcal{Q})$ ramifies also at some prime \mathcal{Q} that is non-split in k the order $\mathcal{O} = \mathcal{O}' \otimes \mathcal{O}_k$ is not maximal even if \mathcal{O}' is maximal.

Example 34. Let $A' = \left(\frac{2,5}{\mathbb{Q}}\right)$ be the quaternion algebra over \mathbb{Q} generated by elements $1, i, j, ij$ such that $i^2 = 2, j^2 = 5$ (and $ij = -ji$). The algebra A' is ramified exactly at the primes 2 and 5. Let \mathcal{O}' be a maximal order in A' . The group $\Gamma_{\mathcal{O}'}(1)$ is a Fuchsian group. Let $\Gamma_{\mathcal{O}'}^B(11)$ be the subgroup corresponding to the Borel subgroup of $PSL_2(\mathbb{F}_{11})$. This subgroup can be interpreted as the group of elements of reduced norm 1 of an Eichler order $\mathcal{E}(11)$ of level 11. The group $\Gamma_{\mathcal{O}'}(1)$ is of index 12 in $\Gamma_{\mathcal{O}'}(1)$ and is torsion-free by Lemma 28, as 5 is split in $\mathbb{Q}(i)$ and 11 is non-split in $\mathbb{Q}(\sqrt{3})$. The genus of the curve $C = \Gamma_{\mathcal{O}'}(11) \backslash \mathbb{H}$ can be easily computed with the already used general volume formula from [Shi63] (see also [Vig80, III, Prop. 2.10]) by which $2 - 2g(C) = -8$. Let $k = \mathbb{Q}(\sqrt{33})$. Then $A = A' \otimes k = \left(\frac{2,5}{k}\right)$ is a quaternion algebra over k and is ramified exactly at the two primes lying over 2. The Eichler order $\mathcal{E}(11)$ in A' is naturally contained in the order $\mathcal{E}(11) \otimes_{\mathbb{Z}} \mathcal{O}_k$ of A and the latter one is contained in $\mathcal{E}(\mathfrak{q}_{11})$ the Eichler order of A corresponding to the prime \mathfrak{q}_{11} of k lying over 11, since the elements $\mathcal{E}(11) \otimes_{\mathbb{Z}} \mathcal{O}_k$ become upper triangular modulo 11, hence modulo \mathfrak{q}_{11} . This gives an embedding of $\Gamma_{\mathcal{O}'}^B(11)$ in $\Gamma_{\mathcal{O}}^B(\mathfrak{q}_{11})$ and hence an embedding of a Shimura curve of genus 5 into the Shimura surface $X = \Gamma_{\mathcal{O}}^B(\mathfrak{q}_{11}) \backslash \mathbb{H} \times \mathbb{H}$ (see Section 4.5) which by construction must be fixed by the involution of second kind on X . This gives a precise characterization of the Shimura surface $Z = X/\sigma$, the quotient of X by the involution on X induced by the involution of second kind:

Proposition 35. *The surface Z is a smooth surface of general type with $p_g = 0$, $K_Z^2 = 4$ and $e(Z) = 8$.*

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